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BOUNDS FOR THE MEAN AND STANDARD DEVIATION
OF LINEAR COMBINATIONS OF ORDER STATISTICS

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Bounds for the mean and standard deviation of linear combinations of order statistics¹⁾ *

by

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SUMMARY

In this paper bounds are given for $E_{\theta} \sum_{s=1}^N \alpha_s O_s$ and $\sigma_{\theta}(\sum_{s=1}^N \alpha_s O_s)$, where $O_1, \dots, O_N$ are the order statistics of $X_1, \dots, X_N$, a random vector with density $p_0(x_1 - \theta_1, \dots, x_N - \theta_N)$ for some $\theta = (\theta_1, \dots, \theta_N)$ and $p_0(x_1, \dots, x_N)$ a completely arbitrary density with respect to Lebesgue measure in R_N . If the X_i are exchangeable under p_0 the bounds depend only on $E_0 \sum_{s=1}^N \alpha_s O_s$ and $\sigma_0(\sum_{s=1}^N \alpha_s O_s)$ plus quantities that can be computed without knowledge of p_0 . The bounds are attainable for some distributions and nearly attainable for some others.

KEY WORDS & PHRASES: *Order statistics, linear combinations of order statistics, bounds for means and standard deviations.*

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1. INTRODUCTION

Let $p_0(x) = p_0(x_1, \dots, x_N)$ be a given density with respect to Lebesgue measure in R_N and consider the family $p_\theta(x) = p_0(x_1 - \theta_1, \dots, x_N - \theta_N)$, $\theta = (\theta_1, \dots, \theta_N) \in R_N$. Let $(X_1, \dots, X_N)$ be a random vector whose density is p_θ for some θ and let $0_1, \dots, 0_N$ be the order statistics of $X_1, \dots, X_N$.

In this paper bounds are given for $E_\theta \sum_{s=1}^N \alpha_s 0_s$ and $\sigma_\theta(\sum_{s=1}^N \alpha_s 0_s)$. These bounds are of the form

$$(1.1) \quad f_1(\theta, \alpha) \leq E_\theta \sum_{s=1}^N \alpha_s 0_s - E_0 \sum_{s=1}^N \alpha_s 0_s \leq f_2(\theta, \alpha)$$

and

$$(1.2) \quad \sigma_\theta\left(\sum_{s=1}^N \alpha_s 0_s\right) \leq \sqrt{\sigma_0^2\left(\sum_{s=1}^N \alpha_s 0_s\right) + f_3(\theta, \alpha) + f_4(\theta, \alpha)},$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$.

The bounds (1.1) require no other assumptions than those above; (1.2) requires a further condition; exchangeability under p_0 of the X_i , for example, is sufficient (see section 3 for the exact condition and also for a bound for $\sigma_\theta(\sum_{s=1}^N \alpha_s 0_s)$ which uses only the assumptions for (1.1)).

With the condition of exchangeability of the X_i under p_0 the functions f_1 , f_2 , f_3 and f_4 can be computed without knowledge of p_0 .

Examples will be given to show that the bounds are in some circumstances exactly attainable and in others almost attainable.

In what follows, section 2 contains the derivation of the bounds for the mean, section 3 the derivation of the bound for the standard deviation. The computation of the functions f_1 , f_2 , f_3 and f_4 when the X_i are exchangeable under p_0 is discussed in section 4. Distributions for which the bounds are attained as well as examples where they are almost attained are given in section 5; section 6 contains some comparisons between actual values and their bounds.

2. BOUNDS FOR THE EXPECTATION OF $\sum_{s=1}^N \alpha_s O_s$

It will be helpful to consider first the expectation of a single order statistic, O_s .

Straightforward computation gives

$$E_{\theta} O_s = \sum_j \int_{T_j} x_{j_s} p_{\theta}(x) dx,$$

where $T_j = \{x_{j_1} \leq \dots \leq x_{j_N}\}$ and $j = \{j_1, \dots, j_N\}$ is a permutation of $\{1, \dots, N\}$. Let $y_r = x_r - \theta_r$, $r = 1, \dots, N$, then

$$E_{\theta} O_s = \sum_j \int_{B_j} (y_{j_s} + \theta_{j_s}) p_0(y) dy$$

with $B_j = \{y_{j_1} + \theta_{j_1} \leq \dots \leq y_{j_N} + \theta_{j_N}\}$. With $A_i = \{y_{i_1} \leq \dots \leq y_{i_N}\}$, $i = \{i_1, \dots, i_N\}$ a permutation of $\{1, \dots, N\}$, we have

$$(2.1) \quad E_{\theta} O_s = \sum_{i,j} \int_{A_i B_j} (y_{j_s} + \theta_{j_s}) p_0(y) dy,$$

where i and j range independently through the $N!$ permutations of $1, \dots, N$.

Concerning the sets $A_i B_j$ we have the following lemma.

LEMMA 2.1. *For each (i,j) there either exists k such that*

$$(i) \quad k \in \{i_s, \dots, i_N\}$$

and

$$(ii) \quad y_{j_s} + \theta_{j_s} \geq y_{i_s} + \theta_k \quad \text{for all } y \in A_i B_j$$

or $A_i B_j$ is empty.

PROOF. Suppose there exists j such that $A_i B_j \neq \emptyset$ and for which there is no k with (i) and (ii). Then for every $k \in \{i_s, \dots, i_N\}$ we have

$$y_{j_s} + \theta_{j_s} < y_{i_s} + \theta_k \quad \text{for some } y \in A_i B_j,$$

which implies, because

$$y_{i_s} \leq y_k \quad \text{for all } k \in \{i_s, \dots, i_N\} \text{ and all } y \in A_{i_j} B_j,$$

that for all $k \in \{i_s, \dots, i_N\}$

$$y_{j_s} + \theta_{j_s} < y_k + \theta_k \quad \text{for some } y \in A_{i_j} B_j.$$

This means that, for all $k \in \{i_s, \dots, i_N\}$, k is strictly to the right of j_s in the j -permutation, but this is a contradiction because $\{i_s, \dots, i_N\}$ contains $N-s+1$ integers and the number of integers strictly to the right of j_s is only $N-s$.

In the same way the following lemma can be proved.

LEMMA 2.2. *For each (i, j) there either exists k such that*

$$(i) \quad k \in \{i_1, \dots, i_s\}$$

and

$$(ii) \quad y_{j_s} + \theta_{j_s} \leq y_{i_s} + \theta_k \quad \text{for all } y \in A_{i_j} B_j$$

or $A_{i_j} B_j$ is empty.

If we now define the random variables whose values on each A_i are

$$(2.2) \quad \begin{cases} \theta_{ls} = \min\{\theta_{i_s}, \dots, \theta_{i_N}\} \\ \theta_{us} = \max\{\theta_{i_1}, \dots, \theta_{i_s}\} \end{cases} \quad s = 1, \dots, N,$$

then we have, on each $A_{i_j} B_j$,

$$(2.3) \quad y_{i_s} + \theta_{ls} \leq y_{j_s} + \theta_{j_s} \leq y_{i_s} + \theta_{us}, \quad s = 1, \dots, N.$$

Integrating with respect to p_0 and using (2.1) gives

$$(2.4) \quad E_0 O_s + E_0 \theta_{\ell s} \leq E_{\theta} O_s \leq E_0 O_s + E_0 \theta_{us} \quad s = 1, \dots, N,$$

where $E_0 \theta_{\ell s} = \sum_i P_0(A_i) \theta_{\ell s}$.

Clearly, for a monotone nondecreasing function $h(O_s)$

$$E_0 h(O_s + \theta_{\ell s}) \leq E_{\theta} h(O_s) \leq E_0 h(O_s + \theta_{us}), \quad s = 1, \dots, N,$$

can be obtained as above. A similar result holds for a function $h(O_1, \dots, O_N)$, where h is monotone in each variable; the upperbound and lowerbound for $E_{\theta} h(O_1, \dots, O_N)$ are determined by the direction of the monotonicity of h in each argument. This result will now be used to find bounds for the expectation of linear combinations $\sum_{s=1}^N \alpha_s O_s$. First note that on $A_i B_j$ we have, for each $s = 1, \dots, N$,

$$(2.5) \quad y_{j_s} + \theta_{j_s} = y_{i_s} + y_{j_s} - y_{i_s} + \theta_{j_s} = y_{i_s} + \Delta_{ijs}, \quad \text{say.}$$

Summing (2.5) with respect to s gives

$$(2.6) \quad \sum_{s=1}^N \Delta_{ijs} = \sum_{s=1}^N \theta_{j_s} = N\bar{\theta}.$$

Hence we can write

$$(2.7) \quad \begin{aligned} \sum_{s=1}^N \alpha_s (y_{j_s} + \theta_{j_s}) &= \sum_{s=1}^N \alpha_s y_{i_s} + \sum_{s=1}^N \alpha_s \Delta_{ijs} = \\ &= \sum_{s=1}^N \alpha_s y_{i_s} + \sum_{s=1}^N (\alpha_s - \alpha_0) \Delta_{ijs} + \alpha_0 N\bar{\theta}, \end{aligned}$$

where α_0 is an arbitrary constant. Since, by (2.3), Δ_{ijs} is in the interval $[\theta_{\ell s}, \theta_{us}]$, we have

$$(2.8) \quad \begin{aligned} \sum_{s=1}^N \alpha_s (y_{j_s} + \theta_{j_s}) &\geq \sum_{s=1}^N \alpha_s y_{i_s} + \sum_{\alpha_s > \alpha_0} (\alpha_s - \alpha_0) \theta_{\ell s} + \\ &\quad + \sum_{\alpha_s < \alpha_0} (\alpha_s - \alpha_0) \theta_{us} + \alpha_0 N\bar{\theta} \end{aligned}$$

and

$$(2.9) \quad \sum_{s=1}^N \alpha_s (y_{j_s} + \theta_{j_s}) \leq \sum_{s=1}^N \alpha_s y_{i_s} + \sum_{\alpha_s > \alpha_0} (\alpha_s - \alpha_0) \theta_{us} + \\ + \sum_{\alpha_s < \alpha_0} (\alpha_s - \alpha_0) \theta_{\ell_s} + \alpha_0 N \bar{\theta}.$$

Integrating with respect to p_0 gives the following lower and upper bounds

$$(2.10) \quad E_{\theta} \sum_{s=1}^N \alpha_s 0_s \geq E_0 \sum_{s=1}^N \alpha_s 0_s + \sum_{\alpha_s > \alpha_0} (\alpha_s - \alpha_0) E_0 \theta_{\ell_s} + \\ + \sum_{\alpha_s < \alpha_0} (\alpha_s - \alpha_0) E_0 \theta_{us} + \alpha_0 N \bar{\theta}$$

and

$$(2.11) \quad E_{\theta} \sum_{s=1}^N \alpha_s 0_s \leq E_0 \sum_{s=1}^N \alpha_s 0_s + \sum_{\alpha_s > \alpha_0} (\alpha_s - \alpha_0) E_0 \theta_{us} + \\ + \sum_{\alpha_s < \alpha_0} (\alpha_s - \alpha_0) E_0 \theta_{\ell_s} + \alpha_0 N \bar{\theta}.$$

The distance between these bounds is

$$(2.12) \quad \sum_{s=1}^N |\alpha_s - \alpha_0| E_0 (\theta_{us} - \theta_{\ell_s}),$$

which is minimized by choosing α_0 as the median of the α_s with respect to the probabilities

$$(2.13) \quad \frac{E_0 (\theta_{us} - \theta_{\ell_s})}{\sum_{s=1}^N E_0 (\theta_{us} - \theta_{\ell_s})}, \quad s = 1, \dots, N.$$

Sometimes a shorter bound can be found by maximizing (2.10) and minimizing (2.11) separately. Consider the upper bound (2.11) which can be written

$$\begin{aligned}
& E_{\theta} \sum_{s=1}^N \alpha_s O_s - E_0 \sum_{s=1}^N \alpha_s O_s \leq \\
& \leq \sum_{s=1}^N \alpha_s E_{\theta} \theta_{us} - \sum_{\alpha_s < \alpha_0} (\alpha_s - \alpha_0) E_0 (\theta_{us} - \theta_{\ell s}) + \alpha_0 [N\bar{\theta} - \sum_{s=1}^N E_0 \theta_{us}] \\
& = \sum_{s=1}^N \alpha_s E_{\theta} \theta_{us} + g(\alpha_0).
\end{aligned}$$

Because $g(\alpha_0)$ is continuous and has a nondecreasing derivative (except for $\alpha_0 = \alpha_s$, $s = 1, \dots, N$) which is nonpositive for small α_0 and nonnegative for large α_0 , $g(\alpha_0)$ either has a minimum or an interval of minima and a minimizing α_0 is an $\alpha_{(s_0)}$ which satisfies the two inequalities

$$\begin{cases} \sum_{\alpha_{(s)} < \alpha_{(s_0)}} E_0 (\theta_{us} - \theta_{\ell s}) \leq \sum_{s=1}^N E_0 \theta_{us} - N\bar{\theta} \\ \sum_{\alpha_{(s)} \leq \alpha_{(s_0)}} E_0 (\theta_{us} - \theta_{\ell s}) \geq \sum_{s=1}^N E_0 \theta_{us} - N\bar{\theta}, \end{cases}$$

where $\alpha_{(1)} \leq \dots \leq \alpha_{(N)}$ are the ordered values of α_s .

In the same way an α_0 maximizing the lower bound for $E_{\theta} \sum_{s=1}^N \alpha_s O_s$ is an $\alpha_{(s_0)}$ which satisfies the two inequalities

$$\begin{cases} \sum_{\alpha_{(s)} < \alpha_{(s_0)}} E_0 (\theta_{us} - \theta_{\ell s}) \leq N\bar{\theta} - \sum_{s=1}^N E_0 \theta_{\ell s} \\ \sum_{\alpha_{(s)} \leq \alpha_{(s_0)}} E_0 (\theta_{us} - \theta_{\ell s}) \geq N\bar{\theta} - \sum_{s=1}^N E_0 \theta_{\ell s}. \end{cases}$$

An example where a shorter bound is obtained by maximizing (2.10) and minimizing (2.11) separately is the following. Let $N = 4$, $\theta_1 = 0 < \theta_2 = \theta_3 = \theta_4 = \theta$ and let the X_i be p_0 -exchangeable. Then, for $\alpha_4 < \alpha_1 < \alpha_3 < \alpha_2$, the median of the α_s with respect to the probabilities (2.13) is α_3 . Using this

median for α_0 in the bounds for $E_{\theta} \sum_{s=1}^N \alpha_s O_s - E_0 \sum_{s=1}^N \alpha_s O_s$ gives

$$\text{lower bound} = \theta \left\{ \frac{3}{4} \alpha_1 + \frac{1}{4} \alpha_2 + \alpha_3 + \alpha_4 \right\}$$

$$\text{upper bound} = \theta \left\{ \alpha_2 + \frac{5}{4} \alpha_3 + \frac{3}{4} \alpha_4 \right\}$$

and the length of this interval is $\theta \left[\frac{3}{4}(\alpha_2 - \alpha_1) + \frac{1}{4}(\alpha_3 - \alpha_4) \right]$. The upper bound is minimized for $\alpha_0 = \alpha_1$ and the upper bound then becomes

$$\theta \left\{ \frac{1}{4} \alpha_1 + \alpha_2 + \alpha_3 + \frac{3}{4} \alpha_4 \right\}.$$

The lower bound is maximized for $\alpha_0 = \alpha_3$ which is the median. The difference between these upper- and lowerbounds is $\theta \left[\frac{3}{4}(\alpha_2 - \alpha_1) + \frac{1}{4}(\alpha_1 - \alpha_4) \right]$ and the difference between the two lengths is $\frac{1}{4}\theta(\alpha_3 - \alpha_1) > 0$.

If $X_1, \dots, X_N$ are p_0 -exchangeable, then $p_0(A_i) = \frac{1}{N!}$, the distributions of θ_{ℓ_s} and θ_{us} do not depend on p_0 and are simple to compute. Some distributions of θ_{ℓ_s} and θ_{us} are given in section 4.

3. UPPERBOUND FOR THE STANDARD DEVIATION OF $\sum_{s=1}^N \alpha_s O_s$.

From (2.8) and (2.9) we have

$$(3.1) \quad \left\{ \begin{array}{l} \sum_{s=1}^N \alpha_s (y_{j_s} + \theta_{j_s}) \text{ is} \\ \leq \sum_{s=1}^N \alpha_s y_{i_s} + \sum_{\alpha_s > \alpha_0} (\alpha_s - \alpha_0) \theta_{us} + \sum_{\alpha_s < \alpha_0} (\alpha_s - \alpha_0) \theta_{\ell_s} + \alpha_0 N \bar{\theta} \text{ and} \\ \geq \sum_{s=1}^N \alpha_s y_{i_s} + \sum_{\alpha_s > \alpha_0} (\alpha_s - \alpha_0) \theta_{\ell_s} + \sum_{\alpha_s < \alpha_0} (\alpha_s - \alpha_0) \theta_{us} + \alpha_0 N \bar{\theta}. \end{array} \right.$$

Then, if

$$Y = \sum_{s=1}^N \alpha_s (y_{j_s} + \theta_{j_s}) - E_{\theta} \sum_{s=1}^N \alpha_s O_s,$$

$$X = \sum_{s=1}^N \alpha_s y_{i_s} - E_0 \sum_{s=1}^N \alpha_s O_s,$$

$$Z_1 = \sum_{\alpha_s > \alpha_0} (\alpha_s - \alpha_0)(\theta_{\ell_s} - E_0 \theta_{\ell_s}) + \sum_{\alpha_s < \alpha_0} (\alpha_s - \alpha_0)(\theta_{us} - E_0 \theta_{us}),$$

$$Z_2 = \sum_{\alpha_s > \alpha_0} (\alpha_s - \alpha_0)(\theta_{us} - E_0 \theta_{us}) + \sum_{\alpha_s < \alpha_0} (\alpha_s - \alpha_0)(\theta_{\ell_s} - E_0 \theta_{\ell_s}),$$

$$A = \alpha_0 N \bar{\theta} - E_\theta \sum_{s=1}^N \alpha_s O_s + E_0 \sum_{s=1}^N \alpha_s O_s + \sum_{\alpha_s > \alpha_0} (\alpha_s - \alpha_0) E_0 \theta_{\ell_s} +$$

$$+ \sum_{\alpha_s < \alpha_0} (\alpha_s - \alpha_0) E_0 \theta_{us},$$

$$B = \alpha_0 N \bar{\theta} - E_\theta \sum_{s=1}^N \alpha_s O_s + E_0 \sum_{s=1}^N \alpha_s O_s + \sum_{\alpha_s > \alpha_0} (\alpha_s - \alpha_0) E_0 \theta_{us} +$$

$$+ \sum_{\alpha_s < \alpha_0} (\alpha_s - \alpha_0) E_0 \theta_{\ell_s},$$

(3.1) becomes

$$(3.2) \quad X + Z_1 + A \leq Y \leq X + Z_2 + B.$$

To find an upper bound for $\sigma(Y) = \sigma(\sum_{s=1}^N \alpha_s O_s)$ consider parabolas

$$k(X + \frac{Z_1 + Z_2 + A + B}{2})^2 + \ell$$

with k and ℓ such that

$$\max\{(X + Z_1 + A)^2, (X + Z_2 + B)^2\} \leq k\{X + \frac{Z_1 + Z_2 + A + B}{2}\}^2 + \ell.$$

With a little algebra one sees that $k > 1$ and

$$\ell \geq \frac{k}{k-1} \left(\frac{Z_2 + B - Z_1 - A}{2} \right)^2$$

are necessary. Let

$$\ell = \frac{k}{k-1} \left(\frac{Z_2 + B - Z_1 - A}{2} \right)^2$$

and k will be chosen after integration. Integrating with respect to p_0 gives

$$(3.3) \quad \sigma_{\theta}^2 \left(\sum_{s=1}^N \alpha_s O_s \right) \leq k E_0 \left(X + \frac{Z_1 + Z_2 + A + B}{2} \right)^2 + \frac{k}{k-1} E_0 \left(\frac{Z_2 + B - Z_1 - A}{2} \right)^2$$

for all $k > 1$.

Minimizing the right hand side of (3.3) with respect to $k > 1$ one finds for the minimizing value of k

$$k = 1 + \left\{ \frac{E_0 \left(\frac{Z_2 + B - Z_1 - A}{2} \right)^2}{E_0 \left(X + \frac{Z_1 + Z_2 + A + B}{2} \right)^2} \right\}^{\frac{1}{2}}.$$

Substitution of this value in (3.3) gives

$$(3.4) \quad \sigma_{\theta}^2 \left(\sum_{s=1}^N \alpha_s O_s \right) \leq \{ E_0 \left(X + \frac{Z_1 + Z_2 + A + B}{2} \right)^2 \}^{\frac{1}{2}} + \{ E_0 \left(\frac{Z_2 + B - Z_1 - A}{2} \right)^2 \}^{\frac{1}{2}}.$$

$$= \{ E_0 \left(X + \frac{Z_1 + Z_2}{2} \right)^2 + \left(\frac{A+B}{2} \right)^2 \}^{\frac{1}{2}} + \{ E_0 \left(\frac{Z_2 + B - Z_1 - A}{2} \right)^2 \}^{\frac{1}{2}}.$$

This bound for $\sigma_{\theta}^2 \left(\sum_{s=1}^N \alpha_s O_s \right)$ contains $E_{\theta} \sum_{s=1}^N \alpha_s O_s$; also, it depends on the correlation between X and $Z_1 + Z_2$. Using (2.10) and (2.11) one obtains

$$\begin{aligned} \frac{|A+B|}{2} &= |\alpha_0 N \bar{\theta} - E_{\theta} \sum_{s=1}^N \alpha_s O_s + E_0 \sum_{s=1}^N \alpha_s O_s + \\ &\quad + \frac{1}{2} \sum_{s=1}^N (\alpha_s - \alpha_0) E_0 (\theta_{us} + \theta_{\ell s})| \\ &\leq \frac{1}{2} \sum_{s=1}^N |\alpha_s - \alpha_0| E_0 (\theta_{us} - \theta_{\ell s}). \end{aligned}$$

Substituting this bound in (3.4) gives the following bound for $\sigma_\theta(\sum_{s=1}^N \alpha_s O_s)$

$$\begin{aligned}
 (3.5) \quad \sigma_\theta(\sum_{s=1}^N \alpha_s O_s) &\leq \\
 &\leq \{\sigma_0^2(\sum_{s=1}^N \alpha_s O_s + \frac{1}{2} \sum_{s=1}^N (\alpha_s - \alpha_0)(\theta_{us} + \theta_{\ell s})) + \\
 &\quad + \frac{1}{4}(\sum_{s=1}^N |\alpha_s - \alpha_0| E_0(\theta_{us} - \theta_{\ell s}))^2\}^{\frac{1}{2}} + \\
 &\quad + [\frac{1}{4} E_0\{\sum_{s=1}^N |\alpha_s - \alpha_0|(\theta_{us} - \theta_{\ell s})\}^2]^{\frac{1}{2}}.
 \end{aligned}$$

If X and $Z_1 + Z_2$ are uncorrelated under p_0 , for example if the X_i are p_0 -exchangeable, the bound (3.5) becomes

$$\begin{aligned}
 (3.6) \quad \sigma_\theta(\sum_{s=1}^N \alpha_s O_s) &\leq \\
 &\leq \{\sigma_0^2(\sum_{s=1}^N \alpha_s O_s) + \frac{1}{4} \sigma_0^2(\sum_{s=1}^N (\alpha_s - \alpha_0)(\theta_{us} + \theta_{\ell s})) \\
 &\quad + \frac{1}{4}(\sum_{s=1}^N |\alpha_s - \alpha_0| E_0(\theta_{us} - \theta_{\ell s}))^2\}^{\frac{1}{2}} + \\
 &\quad + [\frac{1}{4} E_0\{\sum_{s=1}^N |\alpha_s - \alpha_0|(\theta_{us} - \theta_{\ell s})\}^2]^{\frac{1}{2}}.
 \end{aligned}$$

4. THE DISTRIBUTION OF $(\theta_{\ell s}, \theta_{us})$ WHEN THE X_i ARE p_0 -EXCHANGEABLE.

Let $\theta'_1 < \dots < \theta'_M$ be the ordered values of the θ 's, let, for $k = 1, \dots, M$, N_k be the number of $\theta'_j \leq \theta'_k$ and let $N_0 = 0$. If the X_i are p_0 -exchangeable, then $P_0(A_i) = \frac{1}{N!}$ and a somewhat lengthy but straightforward combinatorial argument gives, for the joint distribution of $\theta_{\ell s}$ and θ_{us} ,

$$(4.1) \quad \frac{N!}{(s-1)!(N-s)!} P_0(\theta_{\ell s} = \theta'_j, \theta_{us} = \theta'_k) = \begin{cases} \frac{(N_k - N_{k-1})!}{(s-1-N_{k-1})!(N_k-s)!} & \text{if } j = k \text{ and } N_{k-1}+1 \leq s \leq N_k, \\ A - B - C + D & \text{if } j < k \text{ and } N_{j-1}+1 \leq s \leq N_k, \\ 0 & \\ \text{otherwise,} & \end{cases}$$

where

$$\begin{aligned} A &= \frac{(N_k - N_{j-1})!}{(s-1-N_{j-1})!(N_k-s)!} \\ B &= \begin{cases} \frac{(N_k - N_j)!}{(s-1-N_j)!(N_k-s)!} & \text{if } s \geq N_j + 1 \\ 0 & \text{if not} \end{cases} \\ C &= \begin{cases} \frac{(N_{k-1} - N_{j-1})!}{(s-1-N_{j-1})!(N_{k-1}-s)!} & \text{if } s \leq N_{k-1} \\ 0 & \text{if not} \end{cases} \\ D &= \begin{cases} \frac{(N_{k-1} - N_j)!}{(s-1-N_j)!(N_{k-1}-s)!} & \text{if } N_j + 1 \leq s \leq N_{k-1} \\ 0 & \text{if not.} \end{cases} \end{aligned}$$

From (4.1), or by direct computation, one obtains for the marginal distributions of $\theta_{\ell s}$ and θ_{us}

$$(4.2) \quad \binom{N}{N-s+1} P_0(\theta_{\ell s} = \theta'_k) = \binom{N-N_{k-1}}{N-s+1} - \binom{N-N_k}{N-s+1}$$

and

$$(4.3) \quad \binom{N}{s} P_0(\theta_{us} = \theta'_k) = \binom{N_k}{s} - \binom{N_{k-1}}{s}.$$

If the θ 's are all unequal then $N_k = k$ and the joint and marginal distributions of θ_{ℓ_s} and θ_{us} become

$$(4.4) \quad \frac{N!}{(s-1)!(N-s)!} P_0(\theta_{\ell_s} = \theta'_j, \theta_{us} = \theta'_k) = \begin{cases} (k-s) \binom{k-j-1}{s-j-1} + \binom{k-j}{s-j} \\ \text{if } j \leq s \leq k \\ 0 \\ \text{otherwise,} \end{cases}$$

$$(4.5) \quad P_0(\theta_{\ell_s} = \theta'_k) = \frac{\binom{N-k}{N-s}}{\binom{N}{N-s+1}} = \frac{N-s+1}{s} \frac{\binom{s}{k}}{\binom{N}{k}}$$

and

$$(4.6) \quad P_0(\theta_{us} = \theta'_k) = \frac{\binom{k-1}{s-1}}{\binom{N}{s}} = \frac{s}{N-s+1} \frac{\binom{N-s+1}{N-k+1}}{\binom{N}{N-k+1}}.$$

Two simple limiting distributions for θ_{ℓ_s} , when the θ 's are all unequal, are the following. If $\frac{s}{N} \rightarrow \lambda$, $0 < \lambda < 1$, then

$$(4.7) \quad P_0(\theta_{\ell_s} = \theta'_k) \rightarrow \frac{1-\lambda}{\lambda} \lambda^k, \quad k = 1, 2, \dots$$

If $s = N - a$, a fixed, then

$$(4.8) \quad P_0(\theta_{\ell_s} \leq \theta'_{[Nx]}) \rightarrow (a+1) \int_0^x (1-t)^a dt \quad 0 < x < 1.$$

Similar results hold for the distribution of θ_{us} .

If $M = 2$, i.e. the two sample problem, then the joint distribution of $(\theta_{\ell s}, \theta_{us})$ is

$$(4.9) \quad \begin{cases} P_0(\theta_{\ell s} = \theta_{us} = \theta'_1) = \frac{\binom{n}{s}}{\binom{N}{s}}, \\ P_0(\theta_{\ell s} = \theta'_1, \theta_{us} = \theta'_2) = 1 - \frac{\binom{n}{s}}{\binom{N}{s}} - \frac{\binom{m}{N-s+1}}{\binom{N}{N-s+1}}, \\ P_0(\theta_{\ell s} = \theta_{us} = \theta'_2) = \frac{\binom{m}{N-s+1}}{\binom{N}{N-s+1}}, \end{cases}$$

where $n = N_1$ and $m = N - N_1$. The marginal distributions are

$$(4.10) \quad \begin{cases} P_0(\theta_{\ell s} = \theta'_1) = 1 - \frac{\binom{m}{N-s+1}}{\binom{N}{N-s+1}} \\ P_0(\theta_{\ell s} = \theta'_2) = \frac{\binom{m}{N-s+1}}{\binom{N}{N-s+1}} \end{cases}$$

and

$$(4.11) \quad \begin{cases} P_0(\theta_{us} = \theta'_1) = \frac{\binom{n}{s}}{\binom{N}{s}} \\ P_0(\theta_{us} = \theta'_2) = 1 - \frac{\binom{n}{s}}{\binom{N}{s}}. \end{cases}$$

Note that, for the two sample case, one of the distributions of $\theta_{\ell s}$ and θ_{us} is degenerate, so $\theta_{\ell s}$ and θ_{us} are independent.

5. ATTAINABILITY

In this section distributions are given for which the bounds are attained, as well as examples where they are almost attained.

EXAMPLE a. If p_0 is concentrated on a single A_i , say $i = \{1, \dots, N\}$, and the order of the θ 's is the same as i , here $\theta_1 \leq \dots \leq \theta_N$, then $\theta_{\ell s} = \theta_{us} = \theta_{is}$, here θ_s , with probability 1 for every s and the inequalities (2.10) and (2.11) for the mean and the inequalities (3.5) and (3.6) for the standard deviation become equalities.

EXAMPLE b. If the α_s are all equal, say equal to $\frac{1}{N}$, and α_0 is the common value then the inequalities (2.10), (2.11), (3.5) and (3.6) are equalities.

EXAMPLE c. The inequalities (2.10), (2.11), (3.5) and (3.6) are continuous as the θ 's approach equality and hence can be made arbitrary close to equality for $\theta_{\max} - \theta_{\min}$ sufficiently small.

EXAMPLE d. $\theta_{\max} - \theta_{\min}$ does not have to be close to zero to get approximate equality. For N odd let $X_1, \dots, X_{N+2}$ be independent. For large N there is practically no difference between the distribution of the median of $X_1, \dots, X_{N+2}$ from $U(0,1)$ and the median of $X_1, \dots, X_N$ from $U(0,1)$. Hence, for $\theta > 1$, there is, for large N , practically no difference between the distribution of the median of $X_1, \dots, X_{N+2}$ from $U(0,1)$ and the median of $X_1, \dots, X_{N+2}$ for $X_1, \dots, X_N$ from $U(0,1)$ and X_{N+1}, X_{N+2} from $U(\theta, \theta+1)$ and the inequalities will be almost equalities.

6. SOME COMPARISONS BETWEEN ACTUAL VALUES AND THEIR BOUNDS

Let X_1, X_2, X_3 be independent with X_1 and X_2 double exponential with mean 0 and X_3 double exponential with mean $\theta \geq 0$. Tables 6.1, 6.2 and 6.3 below give, for several values of θ , ($\sigma = \sigma(X_1) = \sqrt{2}$) the values of $E_{\theta} O_s$ and $\sigma_{\theta}(O_s)$, $s = 1, 2, 3$, together with the bounds (2.10), (2.11) and (3.6). For all three tables and all values of θ the value of α_0 used for the bounds is $\alpha_0 = 0$. This value of α_0 maximizes the lower bound (2.10), minimizes the upper bound (2.11) and is the median of the α_s with respect to the weights (2.13).

Table 6.1

Comparison of $E_{\theta}O_1$ and $\sigma_{\theta}(O_1)$ with their bounds

θ	$E_{\theta}O_1$			$\sigma_{\theta}(O_1)$	
	lower bound	value	upper bound	value	upper bound
0	-1.125	-1.125	-1.125	1.1895	1.1895
.04 σ	-1.125	-1.1065	-1.1061	1.1895	1.2060
.1 σ	-1.125	-1.0803	-1.0779	1.1896	1.2310
.4 σ	-1.125	-.9732	-.9364	1.1904	1.3640
σ	-1.125	-.8488	-.6536	1.1932	1.6659

Table 6.2

Comparison of $E_{\theta}O_2$ and $\sigma_{\theta}(O_2)$ with their bounds

θ	$E_{\theta}O_2$			$\sigma_{\theta}(O_2)$	
	lower bound	value	upper bound	value	upper bound
0	0	0	0	.7993	.7993
.04 σ	0	.0189	.0377	.7995	.8227
.1 σ	0	.0471	.0943	.8003	.8591
.4 σ	0	.1838	.3771	.8154	1.0629
σ	0	.4111	.9428	.8852	1.5634

Table 6.3

Comparison of $E_{\theta}O_3$ and $\sigma_{\theta}(O_3)$ with their bounds

θ	$E_{\theta}O_3$			$\sigma_{\theta}(O_3)$	
	lower bound	value	upper bound	value	upper bound
0	1.125	1.125	1.125	1.1895	1.1895
.04 σ	1.1439	1.1443	1.1816	1.1895	1.2128
.1 σ	1.1721	1.1747	1.2664	1.1896	1.2486
.4 σ	1.3136	1.3551	1.6907	1.1920	1.4427
σ	1.5964	1.8520	2.5392	1.2138	1.8996

As a second example, let X_1, X_2, X_3, X_4 be independent with X_1 and $X_2 \sim U(0,1)$ and X_3 and $X_4 \sim U(\theta, \theta+1)$, $0 \leq \theta \leq 1$. Tables 6.4 and 6.5 below give, for several values of θ , ($\sigma = \sigma(X_1) = \frac{1}{\sqrt{12}}$) the values of $E_{\theta}O_1$, $\sigma_{\theta}(O_1)$, $E_{\theta} \frac{O_1+O_4}{2}$ and $\sigma_{\theta}(\frac{O_1+O_4}{2})$. In both tables and for all values of θ the value of α_0 used for the bounds in $\alpha_0 = 0$; this value maximizes the lower bound (2.10), minimizes the upper bound (2.11) and is the median of the α_s with respect to the weights (2.13).

Table 6.4

Comparison of $E_{\theta}O_1$ and $\sigma_{\theta}(O_1)$ with their bounds

θ	$E_{\theta}O_1$			$\sigma_{\theta}(O_1)$	
	lower bound	value	upper bound	value	upper bound
0	.2	.2	.2	.1633	.1633
.04 σ	.2	.2057	.2058	.1634	.1674
.1 σ	.2	.2139	.2144	.1638	.1738
.4 σ	.2	.2494	.2577	.1700	.2091
σ	.2	.2967	.3443	.1937	.2046

Table 6.5

Comparison of $E_{\theta} \frac{{}^0_1+{}^0_4}{2}$ and $\sigma_{\theta} \left(\frac{{}^0_1+{}^0_4}{2} \right)$ with their bounds

θ	$E_{\theta} \frac{{}^0_1+{}^0_4}{2}$			$\sigma_{\theta} \left(\frac{{}^0_1+{}^0_4}{2} \right)$	
	lower bound	value	upper bound	value	upper bound
0	.5	.5	.5	.1291	.1291
.04 σ	.5029	.5058	.5087	.1291	.1329
.1 σ	.5072	.5144	.5217	.1294	.1387
.4 σ	.5289	.5577	.5866	.1327	.1706
σ	.5722	.6443	.7165	.1451	.2468

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